Army Research Laboratory

On the New Evaluation of an Old Integral

by William Walters and Michael Huber

ARL-TR-4689 December 2008

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ARL-TR-4689 **December 2008**

On the New Evaluation of an Old Integral

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REPORT DOCUMENTATION PAGE			*Form Approved* *OMB No. 0704-0188*

1. REPORT DATE *(DD-MM-YYYY)*	2. REPORT TYPE	3. DATES COVERED (From - To)
December 2008	Final	July 2008–December 2008

4. TITLE AND SUBTITLE
On the New Evaluation of an Old Integral

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)
William Walters and Michael Huber*

5d. PROJECT NUMBER
AH80

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
U.S. Army Research Laboratory
ATTN: AMSRD-ARL-WM-TC
Aberdeen Proving Ground, MD 21005-5069

8. PERFORMING ORGANIZATION REPORT NUMBER
ARL-TR-4689

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)

10. SPONSOR/MONITOR'S ACRONYM(S)

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for public release; distribution is unlimited.

13. SUPPLEMENTARY NOTES
*Muhlenberg College, Allentown, PA 18104

14. ABSTRACT
A simple technique is shown to evaluate $\int \frac{dx}{x^m (1+x^n)}$.

15. SUBJECT TERMS
calculus, integration, power series

16. SECURITY CLASSIFICATION OF:			17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
					William Walters
a. REPORT	b. ABSTRACT	c. THIS PAGE			19b. TELEPHONE NUMBER (*Include area code*)
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	UL	12	410-278-6062

Standard Form 298 (Rev. 8/98)
Prescribed by ANSI Std. Z39.18

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1. Introduction

Consider the integral

$$\int \frac{dx}{x^m (1+x)}. \tag{1}$$

In the *CRC Standard Mathematical Tables,*[1] equation 1 can require repeated integral evaluations. Integral 87 shows

$$\int \frac{dx}{x^m (a+bx^n)^{p+1}} = \frac{1}{a}\int \frac{dx}{x^m (a+bx^n)^p} - \frac{b}{a}\int \frac{dx}{x^{m-n} (a+bx^n)^{p+1}}. \tag{2}$$

Enter this integral into your favorite computer algebra system. *Maple* returns

$$\int \frac{dx}{x^m (1+x)} = \frac{x^{-m}(m-1)}{(1-m)m} + \frac{x^{-m}(m-1)\,\mathrm{LerchPhi}(-x,1,-m)}{1-m}, \tag{3}$$

which involves Lerch's Phi transcendent function, defined as

$$\mathrm{LerchPhi}(x,a,m) = \sum_{\mathrm{n=0}}^{\infty} \frac{\mathrm{x}^{\mathrm{n}}}{(\mathrm{m}+\mathrm{n})^{\mathrm{a}}}. \tag{4}$$

Similarly, *Mathematica* gives

$$\int \frac{dx}{x^m (1+x)} = -\frac{x^{1-m}\ \mathrm{Hypergeometric2F1}[1-m, 1, 2-m, x]}{-1+m}, \tag{5}$$

which involves knowing the implementation of Hypergeometric functions. In this report, we seek to provide a simpler evaluation for integrals of this form (equation 1). We state up front that the exponent *m* need not be an integer.

2. Using Infinite Sums

The integrand of equation 1 can be rewritten as follows:

$$\frac{x^{-m}}{1+x}. \tag{6}$$

By long division,

[1] Beyer, W. H., Ed. *CRC Standard Mathematical Tables*. CRC Press, Inc.: West Palm Beach, FL, 1978.

$$\frac{x^{-m}}{1+x} = x^{-m} - x^{-m+1} + x^{-m+2} - x^{-m+3} + \cdots, \tag{7}$$

which is an infinite sum in the form

$$\sum_{i=0}^{\infty} (-1)^i \, x^{-m+i}. \tag{8}$$

Therefore, the integral becomes

$$\begin{aligned} \int \frac{dx}{x^m (1+x)} &= \int \sum_{i=0}^{\infty} (-1)^i \, x^{-m+i} dx \\ &= \sum_{i=0}^{\infty} \int (-1)^i \, x^{-m+i} dx \\ &= \sum_{i=0}^{\infty} \frac{(-1)^i \, x^{-m+i+1}}{-m+i+1} + C, \end{aligned} \tag{9}$$

for some constant C. Noting that

$$\frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i \, x^i \tag{10}$$

converges for $x^2 < 1$ allows us to rewrite equation 9 as

$$\int \frac{dx}{x^m (1+x)} = \frac{1}{x^m} \sum_{i=0}^{\infty} \frac{(-1)^i \, x^{i+1}}{i-m+1} + C; \tag{11}$$

again, m need not be an integer. However, if m = 1, 2, 3, … , then for $i = m–1$, $x^{-m+i} = x^{-1}$, so

$$\int \frac{dx}{x} = \ln x \tag{12}$$

for the $i = m - 1$ term, or

$$\int \frac{dx}{x^m (1+x)} = \frac{1}{x^m} \sum_{i=0}^{m-2} \frac{(-1)^i \, x^{i+1}}{i-m+1} + \ln x + \sum_{i=m}^{\infty} \frac{(-1)^i \, x^{i+1}}{i-m+1} + C. \tag{13}$$

This allows us to sum past the singularity.

3. A Generalization

Generalizing, consider the following:

$$\int \frac{dx}{x^m (1+x^n)} = \int \frac{x^{-m}\, dx}{1+x^n}. \tag{14}$$

From long division, the integrand can be rewritten as $x^{-m}\,(1 - x^n + x^{2n} - x^{3n} + \ldots)$,

or

$$\frac{x^{-m}}{1+x^n} = \sum_{i=0}^{\infty} (-1)^i \; x^{in-m}. \tag{15}$$

Thus,

$$\int \frac{dx}{x^m \left(1+x^n\right)} = \int \sum_{i=0}^{\infty} \left(-1\right)^i x^{in-m} dx = \sum_{i=0}^{\infty} \frac{\left(-1\right)^i \; x^{in-m+1}}{ni - m + 1} + C\,, \tag{16}$$

and we sum prior to integration past any singularity. As before, m and n need not be integers, and this technique converges for | x | < 1.

4. Examples

As a first example, we consider the case with $m = 3$, an integer. We start with equation 2 (shown previously), where $m = 3$, $a = b = n = 1$, and $p = 0$. Continuing, we find that

$$\begin{aligned}
\int \frac{dx}{x^3(1+x)} &= \int \frac{dx}{x^3} - \int \frac{dx}{x^2(1+x)} \\
&= -\frac{x^{-2}}{2} - \left[\int \frac{dx}{x^2} - \int \frac{dx}{x(1+x)} \right] \\
&= -\frac{x^{-2}}{2} + x^{-1} + \ln x - \ln\,(1+x) + C \\
&= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \right] + C \\
&= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots + C\ ,
\end{aligned} \tag{17}$$

which is valid for $-1 < x < 1$. Using our approach and equation 9,

$$\begin{aligned}\int \frac{dx}{x^3(1+x)} &= \int \sum_{i=0}^{\infty} (-1)^i\, x^{-3+i}\, dx \\ &= \int \left[x^{-3} - x^{-2} + x^{-1} - x^0 + x^1 - \cdots \right] dx \\ &= -\frac{1}{2x^2} + \frac{1}{x} + \ln x - x + \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^4}{4} - \cdots + C, \end{aligned} \tag{18}$$

which matches the CRC's solution. When m is a positive integer, equation 2 is adequate. However, as m gets large, the CRC's proposal will involve evaluating several integrals. Our technique is much simpler.

Now, suppose $m = \frac{1}{5}$, $n = 3$, and we keep $a = b = 1$ and $p = 0$. Equation 2 becomes

$$\int \frac{dx}{x^{\frac{1}{5}}(1+x^3)} = \int \frac{dx}{x^{\frac{1}{5}}} - \int \frac{dx}{x^{\frac{-14}{5}}(1+x^3)}\,. \tag{19}$$

While the first integral on the right-hand side is easy enough to evaluate, the second is not trivial, unless the process is repeated or the integrand is expanded in a power series. Turning to equation 15, we find

$$\begin{aligned}\int \frac{dx}{x^{\frac{1}{5}}(1+x^3)} &= \int \sum_{i=0}^{\infty} (-1)^i\, x^{3i-\frac{1}{5}}\, dx \\ &= \int \left[x^{-\frac{1}{5}} - x^{\frac{14}{5}} + x^{\frac{29}{5}} - x^{\frac{44}{5}} + \cdots \right] dx \\ &= \frac{5}{4} x^{\frac{4}{5}} - \frac{5}{19} x^{\frac{19}{5}} + \frac{5}{34} x^{\frac{34}{5}} - \frac{5}{49} x^{\frac{49}{5}} + \cdots + C. \end{aligned} \tag{20}$$

We do not offer any practical applications for the integral. However, the proposed method converges for $x^2 < 1$ or when the independent variable is normalized, to guarantee that the normalized value of x is less than 1 when nondimensionalized; otherwise, the series diverges.

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